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Introduction

In order to determine the elastic compliances of an anisotropic material, it is necessary to conduct deformation experiments on a portion of the material subjected to known surface loads and displacements; In particular, to determine the compliances for thin sheets of circularly orthotropic material it will suffice to load with an eccentric concentrated force, a circular plate of the material supported along its edge. The plate should be formed so that its radii are a system of principal directions for stiffness and circles concentric with the boundary of the plate are the orthogonal set of principal directions of stiffness. Measurements of deflections normal to the plate surface can be made at a suitable number of points appropriately located. The introduction of these measured deflections into the theoretically determined equations for deflection enables one to determine the elastic compliances simply by solving a set of simultaneous equations in the compliances.

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[1]Instructor in Civil Engineering, Rensselaer Polytechnic Institute

[2]Professor of Mechanics, Rensselaer Polytechnic Institute

The theoretical solution for the circularly orthotropic plate with clamped boundary and subjected to an eccentric concentrated force has been obtained by A. M. Sen Gupta [1][3]. However, it turns out that it is more convenient to perform the experiments on a plate having so-called simply-supported or momentless edge. A study of the methods of providing experimentally any desired rotational constraint on a boundary, including the case of momentless edge, has been presented in the technical literature [2]. The usefulness of the momentless edge condition has been particularly emphasized. Consequently, it is desirable to have available the solution of the plate equation which satisfies such a boundary condition. It is the purpose of the present report to provide the solution for that case.

Differential Equation of Plate Flexure

Using s_{ij} to denote the compliances, the stress-strain law for the problem under consideration in polar coordinates is as follows [3,4]:

$$\epsilon_r = s_{11}\,\sigma_r + s_{12}\,\sigma_\theta$$

$$\epsilon_\theta = s_{21}\,\sigma_r + s_{22}\,\sigma_\theta$$

$$\gamma_{r\theta} = s_{66}\,\tau_{r\theta}$$

where $\qquad s_{12} = s_{21}\,.$

[3]Numbers in brackets designate References at end of Report.

Using the principles of plate theory for small deflections and the proposed stress-strain law it can readily be shown that the Differential Equation for flexure is [1,5]:

$$D_r \frac{\partial^4 w}{\partial r^4} + \frac{2D_{r\theta}}{r^2} \frac{\partial^4 w}{\partial r^2 \partial \theta^2} + \frac{D_\theta}{r^4} \frac{\partial^4 w}{\partial \theta^4} + \frac{2D_r}{r} \frac{\partial^3 w}{\partial r^3} - \frac{2D_{r\theta}}{r^3} \frac{\partial^3 w}{\partial r \partial \theta^2} -$$

$$- \frac{D_\theta}{r^2} \frac{\partial^2 w}{\partial r^2} + \frac{2(D_\theta + D_{r\theta})}{r^4} \frac{\partial^2 w}{\partial \theta^2} + \frac{D_\theta}{r^3} \frac{\partial w}{\partial r} = q(r,\theta)$$

where:

w = deflection of the plate

r, θ = polar coordinates

$q(r,\theta)$ = P = concentrated force for this problem

h = thickness of plate

$E_r = 1/s_{22}$

$E_\theta = 1/s_{11}$

$G = 1/s_{66}$

$\nu_r = -s_{12}/s_{22}$

$\nu_\theta = -s_{12}/s_{11}$

$$D_r = \frac{E_r h^3}{12(1-\nu_r \nu_\theta)}$$

$$D_\theta = \frac{E_\theta h^3}{12(1-\nu_r \nu_\theta)}$$

$$D_k = \frac{G h^3}{12}$$

$$D_{r\theta} = 2D_k + \nu_\theta D_r = 2D_k + \nu_r D_\theta$$

Solution of the Plate Equation

Assume that a concentrated force P is applied to the surface of the plate at a distance b from the center. Then imagining the plate of radius a to be divided into two regions by the concentric circle of radius b, we may use the solution for the plate equation with $q(r,\theta)$ taken identically zero for each region of the plate.

The solution for the portion of the plate given in region:

$$0 \leq r \leq b$$

may be written:

$$w_1 = R_0' + \sum_{m=1}^{\infty} R_m' \cos m\theta$$

and the portion given in region:

$$b < r \leq a$$

may be written:

$$w = R_0 + \sum_{m=1}^{\infty} R_m \cos m\theta .$$

Defining

$$k = \sqrt{\frac{D_\theta}{D_r}}$$

the function R_0 may be written:

$$R_0 = A_0 r^{1+k} + B_0 + C_0 r^2 + D_0 r^{1-k}$$

for $k \neq \pm 1$ or 0

and $$R_0 = A_0 + B_0 r^2 + C_0 \log r + D_0 r^2 \log r$$

for $k = \pm 1$.

The characteristic equation corresponding to the homogeneous differential equation is:

$$\lambda^4 - (1 + k^2 + 2\sigma' m^2)\lambda^2 + k^2(m^2 - 1)^2 = 0$$

where $\sigma' = \dfrac{D_{r\theta}}{D_r}$.

The roots of this equation may be written:

$$\pm (p_m + q_m) \quad \text{and} \quad \pm (p_m - q_m) \quad \text{for} \quad m \geq 2$$

and $\pm k_1$ for $m = 1$

whence we have for $m \geq 2$

$$R_m = A_m r^{1+p_m+q_m} + B_m r^{1+p_m-q_m} + C_m r^{1-p_m+q_m} + D_m r^{1-p_m-q_m}$$

and for $m = 1$

$$R_1 = A_1 r^{1+k_1} + B_1 r + C_1 r^{1-k_1} + D_1 r \log r \quad .$$

Similarly we may write solutions for the inner portion of the plate in terms of the primed functions R_0', R_1', R_m'.

Also using the normalizing conditions of boundedness for deflection, moment, and shear force at the center of the plate we have:

$C_0' = D_0' = 0$ from slope and shear

$C_1' = D_1' = 0$ from slope and deflection

$C_m' = D_m' = 0$ from slope and $p_m > q_m$.

Hence

$$R_0' = A_0' r^{1+k} + B_0'$$

$$R_1' = A_1' r^{1+k_1} + B_1' r$$

$$R_m' = A_m' r^{1+p_m+q_m} + B_m' r^{1+p_m-q_m} .$$

Therefore we have six sets of constants to be determined by the two boundary conditions at $r = a$ and the four continuity conditions at $r = b$.

From the requirement of continuity at $r = b$ we have:

$$w = w_1 , \quad \frac{\partial w}{\partial r} = \frac{\partial w_1}{\partial r} , \quad \text{and} \quad \frac{\partial^2 w}{\partial r^2} = \frac{\partial^2 w_1}{\partial r^2} .$$

Also, if the concentrated force P is expanded in a Fourier series in the angle θ we have:

$$P = \frac{P}{\pi b} \left(\frac{1}{2} + \sum_{m=1}^{\infty} \cos m \theta \right) .$$

Furthermore, the shear force condition at $r = b$ is:

$$\frac{\partial^3 w}{\partial r^3} - \frac{\partial^3 w_1}{\partial r^3} = \frac{P}{\pi b D_r} \left(\frac{1}{2} + \sum_{m=1}^{\infty} \cos m \theta \right)$$

and the boundary conditions at the edge of the plate, $r = a$, for so-called simple support or momentless boundary are:

$$w = 0$$

and $$M_r = \frac{\partial^2 w}{\partial r^2} + \nu_\theta \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) = 0$$

at $r = a$

but since $w = 0$ at $r = a$

then $\frac{\partial^2 w}{\partial \theta^2} = 0$ at $r = a$

and the $M_r = 0$ condition may be simply written:

$$\frac{\partial^2 w}{\partial r^2} + \frac{\nu_\theta}{r} \frac{\partial w}{\partial r} = 0 .$$

Using the six sets of equations to determine the corresponding six sets of constants the primed and unprimed functions R become explicitly:

$$R_0 = \frac{P a^2}{4\pi D_r (k^2-1)} \left\{ \frac{1}{1+k} \left[\frac{2(1+\nu_\theta)}{(k+\nu_\theta)} - \frac{k-1}{k} \frac{\nu_\theta - k}{\nu_\theta + k} \left(\frac{b}{a}\right)^{1+k} \right] \left(\frac{r}{a}\right)^{1+k} + \right.$$

$$\left. + \frac{k-1}{k+1} \frac{(2+k+\nu_\theta)}{(k+\nu_\theta)} + \frac{2}{k+1} \frac{1+\nu_\theta}{k+\nu_\theta} \left(\frac{b}{a}\right)^{1+k} - \left(\frac{r}{a}\right)^2 - \frac{1}{k} \left(\frac{b}{a}\right)^2 \left(\frac{b}{r}\right)^{k-1} \right\}$$

$$R_1 = - \frac{P b^{k_1+1}}{2\pi k_1^3 D_r} \left\{ \frac{1}{r^{k_1-1}} - \frac{2a^{k_1}(1+\nu_\theta) - (1+\nu_\theta - k_1) b^{k_1} r^{1+k_1}}{(a^{2k_1} \quad b^{k_1})(1+k_1+\nu_\theta)} + \right.$$

$$\left. + \frac{2(1+\nu_\theta)}{1+k_1+\nu_\theta} \frac{a^{k_1} - b^{k_1}}{a^{k_1} b^{k_1}} r - \frac{2k_1 r}{b^{k_1}} \log \frac{a}{r} \right\}$$

$$R_m = \frac{Pb^2}{8\pi p_m q_m (p_m - q_m) D_r} \left\{ \frac{p_m - q_m}{q_m(1+2p_m+\nu_\theta)} \left[(1+\nu_\theta)\left(\frac{b}{a}\right)^{p_m-q_m-1} - \frac{p_m(1+\nu_\theta-2q_m)}{p_m+q_m}\left(\frac{b}{a}\right)^{p_m+q_m-1} \right] \left(\frac{r}{a}\right)^{p_m+q_m+1} + \right.$$

$$+ \left[\frac{(p_m-q_m)(1+\nu_\theta)}{q_m(1+2p_m+\nu_\theta)}\left(\frac{b}{a}\right)^{p_m+q_m-1} - \frac{p_m(1+2q_m+\nu_\theta)}{q_m(1+2p_m+\nu_\theta)}\left(\frac{b}{a}\right)^{p_m-q_m-1} \right] \left(\frac{r}{a}\right)^{p_m-q_m+1} +$$

$$\left. + \left(\frac{b}{r}\right)^{p_m-q_m-1} - \frac{p_m - q_m}{p_m + q_m}\left(\frac{b}{r}\right)^{p_m+q_m-1} \right\}$$

and for the primed functions:

$$R_0' = \frac{P a^2}{4\pi D_r(k^2-1)} \left\{ \frac{1}{1+k} \left[\frac{2(1+\nu_\theta)}{(k+\nu_\theta)} - \frac{k-1}{k} \frac{\nu_\theta - k}{\nu_\theta + k} \left(\frac{b}{a}\right)^{1+k} \right] \left(\frac{r}{a}\right)^{1+k} + \right.$$

$$\left. + \frac{k-1}{k+1} \cdot \frac{2+k+\nu_\theta}{k+\nu_\theta} + \frac{2}{k+1} \frac{1+\nu_\theta}{k+\nu_\theta} \left(\frac{b}{a}\right)^{1+k} - \left(\frac{b}{a}\right)^2 - \frac{1}{k}\left(\frac{b}{a}\right)^2 \left(\frac{r}{b}\right)^{1+k} \right\}$$

$$R_1' = \frac{-\ Pb^{k_1+1}}{2\pi k_1^3 D_r} \left[\frac{r^{1+k_1}}{b^{2k_1}} - \frac{2a^{k_1}(1+\nu_\theta) - (1+\nu_\theta - k_1) b^{k_1}}{(1+k_1+\nu_\theta)(a^{2k_1} b^{k_1})} r^{1+k_1} + \right.$$

$$\left. + \frac{2(1+\nu_\theta)}{(1+k_1+\nu_\theta)} \frac{a^{k_1} - b^{k_1}}{a^{k_1} b^{k_1}} r - \frac{2k_1 r}{b^{k_1}} \log \frac{a}{b} \right]$$

$$R_m' = \frac{Pb^2}{8\pi p_m q_m (p_m - q_m) D_r} \left\{ \frac{p_m - q_m}{q_m(1+2p_m+\nu_\theta)} \left[(1+\nu_\theta)\left(\frac{b}{a}\right)^{p_m-q_m-1} - \frac{p_m(1+\nu_\theta-2q_m)}{q_m(p_m+q_m)} \left(\frac{b}{a}\right)^{-1+p_m+q_m} \right] \left(\frac{r}{a}\right)^{p_m+q_m+1} + \right.$$

$$+ \left[\frac{(p_m-q_m)(1+\nu_\theta)}{q_m(1+2p_m+\nu_\theta)} \left(\frac{b}{a}\right)^{p_m+q_m-1} - \frac{p_m(1+\nu_\theta+2q_m)}{q_m(1+\nu_\theta+2p_m)} \left(\frac{b}{a}\right)^{p_m-q_m-1} \right] \left(\frac{r}{a}\right)^{p_m-q_m+1}$$

$$\left. - \frac{(p_m-q_m)}{p_m+q_m} \left(\frac{r}{b}\right)^{p_m+q_m+1} + \left(\frac{r}{b}\right)^{p_m-q_m+1} \right\}$$

It can be easily shown that the isotropic solution will be obtained if we use the following values for the parameters:

$$k = 1 \ , \quad k_1 = 2 \ , \quad p_m = m \ , \quad q_m = 1 \ ,$$

$$D_r = D \ , \quad \text{and} \quad \nu_\theta = \nu \ .$$

Also,

$$\lim_{b \to 0} \left. w \right|_{r=b} = \frac{P a^2}{4\pi D_r (k+1)^2} \cdot \frac{2+k+\nu_\theta}{k+\nu_\theta}$$

which is the deflection under a concentrated load P of a simply supported circular plate with circular anisotropy [5].

Now that the R functions are explicitly given in terms of the elastic compliances the deflection equation for a simply supported circular plate with eccentric concentrated force is completely determined.

Experiments to determine deflections can now be conducted on the anisotropic plate with the given load and boundary conditions. As a consequence, a sufficient number of simultaneous equations involving only the elastic compliances as unknowns can be set up and solved to give the desired value of the compliances.

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